On cellular covers with free kernels

José L. Rodríguez and Lutz Strüngmann

January 7th, 2010

Abstract

Recall that a homomorphism of R-modules $\pi : G \to H$ is called a *cellular cover* over H if π induces an isomorphism $\pi_* : \mathrm{Hom}_R(G, G) \cong \mathrm{Hom}_R(G, H)$, where $\pi_*(\varphi) = \pi\varphi$ for each $\varphi \in \mathrm{Hom}_R(G, G)$ (where maps are acting on the left). In this paper we show that every cotorsion-free module K of finite rank can be realized as the kernel of a cellular cover of some cotorsion-free module of rank 2. In particular, every free abelian group of any finite rank appears then as the kernel of a cellular cover of a cotorsion-free abelian group of rank 2. This situation is best possible in the sense that cotorsion-free abelian groups of rank 1 do not admit cellular covers with free kernel except for the trivial ones. This work comes motivated by an example due to Buckner and Dugas, and recent results obtained by Göbel–Rodríguez–Strüngmann, and Fuchs–Göbel.

1 Introduction

Recently cellular covers of groups and modules have attracted much attention in the literature. Recall that a homomorphism $\pi : G \to H$ of groups is a *cellular cover* over H, if every homomorphism $\varphi : G \to H$ lifts uniquely to an endomorphism $\widetilde{\varphi}$ of G such that $\pi\widetilde{\varphi} = \varphi$. In case that $\pi : G \to H$ is an epimorphism we say that

$$1 \to K \to G \xrightarrow{\pi} H \to 1$$

is a *cellular exact sequence*. Cellular covers are the algebraic counterpart of cellular approximations of topological spaces in the sense of J.H.C. Whitehead, or the more general cellularization maps extensively studied in homotopy theory in the 90s (see e.g. [1], [5], [7], [15], [16]). As for the dual case, namely for localizations, there is sometimes a good interplay between cellularization of spaces and cellularization of groups and modules. This motivated a careful study of the algebraic setting (see e.g. [3], [4], [9], [17], [18]).

The general goal is to completely classify (up to isomorphism) all possible cellular exact sequences for fixed cokernel H or kernel K. In some cases this is possible. For instance, if H is divisible, then G (above) can be determined explicitly as was shown in [6]. And if H is torsion and reduced, then the cellular exact sequence collapses and $K = 0$, see [11].

However, if H is reduced (and torsion-free), then K must be cotorsion-free, see [2], [8], [11]

[0]The first author was supported by the Spanish Ministry of Education and Science MEC-FEDER grant MTM2007-63277.

The second author was supported by the project No. 963-98.6/2007 of the German-Israeli Foundation for Scientific Research & Development.

Subject classification (2000):

Primary: 20K20, 20K30; Secondary: 16S60, 16W20.

Key words and phrases: cellular cover, co-localization, cotorsion-free, free abelian group

and a result by Buckner and Dugas [2] shows that any cotorsion-free module K is the kernel of arbitrarily large cellular covers G of size $\geq 2^{\aleph_0}$. Dually, it was shown in [14] that any cotorsion-free module H of size $\geq 2^{\aleph_0}$ satisfying natural rigidity conditions is the cokernel of arbitrarily large cellular covers. This makes a classification impossible.

Also the case of finite rank modules was treated in [14] and also in [11]. If R is a subgroup of the rational numbers and R is a ring, then R does not admit cellular covers except for the trivial ones (see [11]). However, if R is not a ring, then R has arbitrarily large cellular covers by [11]. Unfortunately, the proof in [11] of this result contains a gap that we will fix in this paper (see Section 1). Moreover, the construction in [11] implies that the kernels of the cellular covers in this situation are p-divisible for some prime p and we will prove that this is justified by showing the following (see Theorem 3.1): If R is a subgroup of $\mathbb{Q}$, then R does not admit cellular covers with free kernel. Nevertheless, we prove in Section 3 that any cotorsion-free module K of finite rank is in fact the kernel of a cellular cover with cokernel of rank 2 (see Theorem 4.2). That the rank of K has to be countable is a necessary restriction by the following result. If K is free, then $|K| \leq |H|$ ([8]). This clarifies the situation for cellular covers with cokernel of rank 1 and 2 almost completely since it is left as an open question if also a free module of countably infinite rank can be the kernel of some cellular cover of cokernel of rank 2.

2 Cellular covers of rational groups

In this section we consider a result from [11] and fill a gap in its proof. Recall from [8, Theorem 2.11] that there are torsion-free abelian groups L that admit arbitrarily large cellular covers

$$0 \to K \to G \to L \to 0.$$

Necessarily, the kernel K of the cellular cover then has to be non-free if $|K| > |L|$ by [8, Proposition 1.4]. This result was strengthened in [11] (to groups L of rank one) where the following was claimed. Recall that any rank one group S is a subgroup of the rational numbers $\mathbb{Q}$ and is of the form $S = \left\langle \frac{1}{p^{n_p}} : n_p < m_p, p \in \Pi \right\rangle$ for some natural numbers or the ∞-symbol $m_p \in \mathbb{N}_0 \cup \{\infty\}$ where Π is the set of all primes and $\mathbb{N}_0$ is the set of natural numbers including 0. The type of S is then the equivalence class (in the sense of Baer) of the height sequence $(m_p - 1 : p \in \Pi)$ (with the convention that $\infty - 1 = \infty$) where two sequences $(m_p - 1 : p \in \Pi)$ and $(k_p - 1 : p \in \Pi)$ are equivalent if the sum $\sum\limits_{p \in \Pi} m_p - k_p$ is finite. The types form a lattice that has a rich structure (see [10, Chapter XIII]).

Theorem 2.1. *Let $R \subseteq \mathbb{Q}$ be a torsion-free abelian group of rank 1.*

(I) Suppose that R is not a ring. Then R admits a cellular cover of rank κ for every cardinal $\kappa \geq 1$ ([11, Lemma 5.3]);

(II) Suppose R is a ring. Then the only cellular covers of R are the trivial ones ([11, Theorem 6.1]).

The idea of the proof for (I) is the following (see [11, Proof of Lemma 5.3]). For simplicity let $R = \left\langle \frac{1}{p} : q \neq p \in \Pi \right\rangle$ for some fixed prime q. The general case follows with easy modifications. For a cardinal $\kappa \geq 1$ choose a torsion-free group K of rank κ such that $\mathrm{End}(K) \cong \mathbb{Z}[\frac{1}{q}]$

and K does not contain any pure subgroup of rank one whose type is at least the type of R. The authors then choose elements $b_p \in K$ for $p \neq q$ such that b_p is not divisible by p in K and put $G := \left\langle K, \frac{a+b_p}{p} : q \neq p \in \Pi \right\rangle$ and claim that

$$(*) \quad 0 \to K \to G \xrightarrow{\pi} R \to 0$$

is a cellular cover with the obvious map π sending a onto $1 \in R$ and K to 0. One easily checks that

- $\mathrm{End}(R) \cong \mathbb{Z}$
- K is fully invariant in G as it is the maximal q-divisible subgroup of G
- $\mathrm{Hom}(K, R) = 0$ by the q-divisibility of K.

Moreover, the authors also claim that

- $\mathrm{Hom}(G, K) = 0$

which then would imply that the above sequence $(*)$ is indeed a cellular cover by [8, Lemma 3.5].
However, the condition on the elements b_p is not strong enough to force $\mathrm{Hom}(G, K) = 0$ as the following Lemma shows.

Lemma 2.2. *If $K = \mathbb{Z}[\frac{1}{q}]$ and $b_p = p - 1 \in K$, then the sequence $(*)$ is not a cellular cover of R. However, $b_p \notin pK$ for every $q \neq p \in \Pi$.*

Proof. It is easy to see that in this case the sequence $(*)$ splits since $\frac{a+b_p}{p} = \frac{a+p-1}{p} \equiv \frac{a-1}{p}$ mod K and so $G = \left\langle K, \frac{a-1}{p} \right\rangle \cong K \oplus R$. Hence $(*)$ cannot be a cellular cover. □

This shows that one has to choose the elements b_p more carefully; we now show how.

Proof. (of Theorem 2.1 (I)) As above let $R = \left\langle \frac{1}{p} : q \neq p \in \Pi \right\rangle$ for some fixed prime q. For a cardinal $\kappa \geq 1$ choose a torsion-free group K of rank κ such that $\mathrm{End}(K) \cong \mathbb{Z}[\frac{1}{q}]$ and K is homogeneous of type $\mathbb{Z}[\frac{1}{q}]$. Such a group exists for every cardinal κ as was pointed out in [11] (see also [12]). We now divide the set of primes Π into two disjoint infinite subsets Π_1 and Π_2 with $q \in \Pi_1$. Fix a bijection $\sigma : \mathbb{Z}[\frac{1}{q}] \to \Pi_2$ such that $z \notin \sigma(z)\mathbb{Z}[\frac{1}{q}]$ for all $0 \neq z \in \mathbb{Z}[\frac{1}{q}]$. By the type condition on K we may now pick elements $b_{\sigma(z)} \in K$ such that $zb_{\sigma(z)} \notin \sigma(z)K$ for all $0 \neq z \in \mathbb{Z}[\frac{1}{q}]$. Let

$$G = \left\langle K, \frac{a}{p}, \frac{a + b_r}{r} : p \in \Pi_1, r \in \Pi_2 \right\rangle.$$

As above it is easy to see that K is fully invariant in G and that $\mathrm{Hom}(K, R) = 0$. We claim that now also $\mathrm{Hom}(G, K) = 0$. Therefore assume $\varphi \in \mathrm{Hom}(G, K)$. Then $\varphi \restriction_K = z \cdot id_K$ for some $z \in \mathbb{Z}[\frac{1}{q}]$ by the full invariance of K. Moreover, $\varphi(a) = k \in K$. We obtain

$$\varphi(\frac{a}{p}) = \frac{k}{p} \in K$$

for all $p \in \Pi_1$. But K does not contain non-zero elements of type other than $\mathbb{Z}[\frac{1}{q}]$, hence $\varphi(a) = 0$. Now

$$\varphi(\frac{a + b_r}{r}) = \frac{zb_r}{r} \in K$$

for all $r \in \Pi_2$. However, by the choice of σ we have

$$\frac{zb_{\sigma(z)}}{\sigma(z)} \notin K$$

a contradiction if $z \neq 0$. Thus $z = 0$ and hence $\mathrm{Hom}(G, K) = 0$ and the sequence

$$0 \to K \to G \to R \to 0$$

is a cellular cover by [8, Lemma 3.5]. □

3 Cellular covers of rational groups with free kernel

In the previous section it was shown that there is a cellular cover

$$0 \to K \to G \to R \to 0$$

of a rational group R such that the kernel K is isomorphic to the rational group $\mathbb{Z}[\frac{1}{q}]$ for some fixed prime q. We now ask if one can get the same result replacing $\mathbb{Z}[\frac{1}{q}]$ by the integers $\mathbb{Z}$ or more generally by a free group and still have a cokernel of rank one. The following result shows that this is impossible.

Theorem 3.1. *Let $H \subseteq \mathbb{Q}$ be a torsion-free abelian group of rank 1 and H_0 its nucleus. Suppose that H is not a ring. Then H does not admit any cellular cover with kernel a free H_0-module.*

Proof. Let H be as stated. Then $H \ncong H_0$ and any cellular cover of H is an H_0-module by [8]. Without loss of generality we may assume that $H_0 = \mathbb{Z}$ and that $H = \langle \frac{1}{p} : p \in \Pi \backslash \{2\} \rangle$. The general case is obtained by simple modification of our arguments and therefore left to the reader.

Let $F = \langle e_i : i \in I \rangle$ be a free-abelian group, with I a non-empty index-set, and suppose that there exists a cellular exact sequence

$$0 \to F \longrightarrow G \xrightarrow{\pi} H \to 0, \tag{3.1}$$

that is, $\mathrm{Hom}(G, F) = 0$ and every homomorphism $\varphi : G \to H$ lifts to a (unique) endomorphism $\psi : G \to G$ such that $\pi\psi = \varphi$.

If G fits in (3.1) then there exist elements $a \in G$ and $z_p \in F$ for all $p \in \Pi$ such that:

$$G = \langle F, \frac{a + z_p}{p} : p \in \Pi \rangle$$

where $\pi(a) = 1 \in H$ and $\pi(\frac{a+z_p}{p}) = \frac{1}{p} \in H$. We express each z_p as a linear combination of the base elements of F, say $z_p = \sum_{i \in I} z_p^i e_i$, where almost all the coefficients are equal to 0. We can assume without loss of generality that every non-trivial coefficient z_p^i satisfies $(z_p^i, p) = 1$, otherwise we could just erase it, as the term $\frac{z_p^i e_i}{p}$ would belong to F.

We can now give explicit generators of $\mathrm{Hom}(G, H)$. For every $r, s \in \mathbb{Z}$, and $i \in I$ define the homomorphism:

$$\varphi = \varphi_{s,r,i} : G \to H \quad \text{by} \quad \varphi(a) = r, \quad \varphi(e_j) = 0 \quad \text{for } j \neq i, \quad \varphi(e_i) = s;$$

Note that φ is well defined since $(r + z_p^i s)$ is an integer and therefore $(r + z_p^i s) \in pH$, for all $p \in \Pi \backslash \{2\}$ and $i \in I$. By linearity we then have

$$\varphi\left(\frac{a + z_p}{p}\right) = \frac{r + z_p^i s}{p}.$$

Now, if $q \in \Pi$ is a fixed prime such that $r + z_q^i s \in q\mathbb{Z}$, then one can define

$$\varphi' = \varphi_{r,s,i}^q : G \to H \quad \text{by} \quad \varphi'(a) = \frac{r}{q}, \quad \varphi'(e_j) = 0 \quad \text{for } j \neq i, \quad \varphi'(e_i) = \frac{s}{q},$$

and therefore

$$\varphi'\left(\frac{a + z_p}{p}\right) = \frac{r + s z_p^i}{pq}.$$

Note that the condition $r + z_q^i s \in q\mathbb{Z}$ is needed since otherwise $\frac{r + s z_q^i}{q^2} \notin H$. By assumption we know that (3.1) is a cellular sequence, hence there exist unique endomorphisms of G, $\psi = \psi_{r,s,i}$ and $\psi' = \psi_{r,s,i}^q$ such that $\pi\psi = \varphi$ and $\pi\psi' = \varphi'$. Torsion-freeness of H and uniqueness of lifting imply $q\psi' = \psi$.

We want to say more about the action of ψ. Obviously,

$$\varphi_{r,s,i} = \varphi_{r,0,i} + \varphi_{0,s,i} = r\varphi_{1,0,i} + s\varphi_{0,1,i}.$$

If $s = 0$, then the unique lifting of $\varphi_{r,0,i}$ is multiplication by r on G. Similarly, uniqueness implies that $s\psi_{0,1,i} = \psi_{0,s,i}$. It is now easy to check that there exist elements h_i, $h_a \in F$ such that ψ is given by:

$$\begin{array}{llll} \psi : & G & \longrightarrow & G \\ & a & \mapsto & ra + sh_a \\ & e_j & \mapsto & re_j + sh_j \quad \text{for } j \neq i \\ & e_i & \mapsto & re_i + sh_i + sa \\ & \frac{a+z_p}{p} & \mapsto & \psi(\frac{a+z_p}{p}) \end{array} \tag{3.2}$$

Note that $q\psi' = \psi$ yields in particular

$$\psi(e_i) = re_i + sh_i + sa \in qG \tag{3.3}$$

Since $\psi(a + z_p) \in pG$, we conclude for all $p \in \Pi$

$$\begin{aligned} \psi(a + z_p) &= ra + sh_a + \sum_{j \neq i} z_p^j (re_j + sh_j) + z_p^i (re_i + sh_i + sa) \\ &= r(a + z_p) + s(h_a + \sum_j z_p^j h_j + z_p^i a) \in pG. \end{aligned} \tag{3.4}$$

Using that $(a + z_p) \in pG$, we then obtain

$$s(h_a + \sum_j z_p^j h_j + z_p^i a) \in pG.$$

Subtracting $sz_p^i(a+z_p) \in pG$ from the previous expression we get

$$s(h_a + \sum_j z_p^j h_j - z_p^i z_p) \in pG \cap F = pF \tag{3.5}$$

since H is torsion-free and hence F is pure in G.

Now recall that $\psi = q\psi'$ for the fixed prime q satisfying $(r + sz_q^i) \in q\mathbb{Z}$. Hence $\psi(a+z_p) \in qG$ for all primes $p \in \Pi$. From (3.4) and adding and subtracting $sz_q^i(a+z_p)$ we obtain

$$\begin{aligned}
\psi(a+z_p) &= \psi(a+z_p) + sz_q^i(a+z_p) - sz_q^i(a+z_p) \\
&= (r+sz_q^i)(a+z_p) + s[h_a + \sum_j z_p^j h_j + z_p^i a] - sz_q^i(a+z_p) \\
&= (r+sz_q^i)(a+z_p) + s[h_a + \sum_j z_p^j h_j + z_p^i a - z_q^i a - z_q^i z_p] \in qG.
\end{aligned}$$

Since $(r + sz_q^i) \in q\mathbb{Z}$ we deduce from this equation:

$$s[h_a + \sum_j z_p^j h_j + z_p^i a - z_q^i a - z_q^i z_p] \in qG.$$

But (3.5) for $p = q$ tells us that $s(h_a + \sum_j z_q^j h_j - z_q^i z_q) \in qG$, hence

$$s[\sum_j (z_p^j - z_q^j)h_j + (z_p^i - z_q^i)a + z_q^i(z_q - z_p)] \in qG.$$

In the last expression $(z_p^i - z_q^i)a \notin F$, thus subtracting $(z_p^i - z_q^i)(a + z_q) \in qG$ we get

$$s\left[\sum_j (z_p^j - z_q^j)h_j + z_q^i(z_q - z_p) - (z_p^i - z_q^i)z_q\right] \in qG \cap F = qF. \tag{3.6}$$

Now we need to look at the exact presentation of the elements h_j in F. Let $h_j = \sum_k h_j^k e_k$, where almost all coefficients h_j^k are equal to 0. By (3.2) and the fact that $\psi = q\psi'$ we conclude that $sh_j^i \in q\mathbb{Z}$ for all $j \neq i$ when restricting to the i^{th}-component $\mathbb{Z}e_i$ of F. Restricting (3.6) to the i^{th}-component of F we thus obtain

$$s[(z_p^i - z_q^i)h_i^i + (z_q^i - z_p^i)z_q^i - (z_p^i - z_q^i)z_q^i)] \in q\mathbb{Z}.$$

Therefore

$$s(z_p^i - z_q^i)(h_i^i - z_q^i - z_q^i) = s(z_p^i - z_q^i)(h_i^i - 2z_q^i) \in q\mathbb{Z}.$$

Adding $2(z_p^i - z_q^i)(r + sz_q^i) \in q\mathbb{Z}$ to this expression we obtain:

$$(z_p^i - z_q^i)(sh_i^i + 2r) \in q\mathbb{Z}. \tag{3.7}$$

From (3.3) we have $re_i + sh_i + sa \in qG$, and also $s(a + z_q) \in qG$, thus their difference

$$(re_i + sh_i + sa) - (sa + sz_q) = re_i + sh_i - sz_q \in qG \cap F = qF.$$

Restricted to the i^{th}-component gives:

$$r + sh_i^i + sz_q^i \in q\mathbb{Z}. \tag{3.8}$$

The difference between the expression (3.7) and $(z_p^i - z_q^i)$ times the expression in (3.8) gives

$$(z_p^i - z_q^i)(r - sz_q^i) \in q\mathbb{Z}.$$

Again, $(r + sz_q^i) \in q\mathbb{Z}$ implies

$$(z_p^i - z_q^i)(2r) \in q\mathbb{Z}. \tag{3.9}$$

Note that equation (3.9) holds for any pair of primes p, q and integers r, s such that $(r + sz_q^i) \in q\mathbb{Z}$.
We now distinguish two cases and make particular choices for the data $p.q, r, s$.

Case 1: *There exists an index i and primes p, q such that $z_p^i = 0$ and $z_q^i \neq 0$.*
Note that in this case $z_q^i \notin q\mathbb{Z}$ and recall that q must be different from 2. We choose integers r, s relatively prime to q such that $(r + sz_q^i) \in q\mathbb{Z}$. Thus equation (3.9) yields that $2rz_q^i \in q\mathbb{Z}$ - a contradiction.

Case 2: *Not case* 1.
Then for every index i either $z_p^i = 0$ for all primes p or $z_p^i \neq 0$ for all primes p. In the latter we fix i and claim that

Either there exists a prime $q(\neq 2)$ such that $(z_p^i - z_q^i) \notin q\mathbb{Z}$, for some $p \in \Pi\backslash\{2\}$ or $z_p^i = z_q^i$ for all p and q.

Suppose that $(z_p^i - z_q^i) \in q\mathbb{Z}$ for all $p \in \Pi$. Then obviously $(z_p^i - z_l^i) \in q\mathbb{Z}$ for all primes $p, q, l \in \Pi$. This is impossible once we fix two primes p and l. Therefore, the only possibility is that $z_p^i = z_l^i$ for all primes p and l in Π.

Now assume that there exists a prime $q(\neq 2)$ such that $(z_p^i - z_q^i) \notin q\mathbb{Z}$, for some $p \in \Pi\backslash\{2\}$. As above we may choose integers r, s relatively prime to q such that $(r + sz_q^i) \in q\mathbb{Z}$. Thus equation (3.9) yields that $2r(z_p^i - z_q^i) \in q\mathbb{Z}$ - a contradiction.

The only possibility left is that for all i we deduce that $z_p^i = z_l^i$ for all $p, l \in \Pi$. However, this implies that $z_p = z_q$ for all primes p, q and hence the group $G = \langle F, \frac{a+z}{p} : p \in \Pi\backslash\{2\}\rangle$, for a fixed element $z \in F$. In particular, one can define a section $H \to G$ of the cellular cover $G \to H$, given by $1 \mapsto a + z$, which is not possible. This finishes the proof of the theorem. $\square$

4 Cellular covers with cotorsion-free kernels

Inspired by the previous sections we are now interested in the following question: Can we realize every cotorsion-free abelian group K (in particular every free abelian group K) as the kernel of a cellular cover of some torsion-free group of rank two? By Theorem 3.1 this is the best we can hope for and by [8, Proposition 1.4] we will have to assume that K is countable since free groups are cotorsion-free.

4.1 Notation (see [12])

Let R be a commutative ring with 1 and a distinguished countable multiplicatively closed subset $\mathbb{S} = \{s_n : n \in \omega\}$ such that R is $\mathbb{S}$-reduced and $\mathbb{S}$-torsion-free. Thus $\mathbb{S}$ induces a Hausdorff topology on R, taking $q_m R$ $(m \in \mathbb{Z})$ as the neighborhoods of zero where $q_m =$

$\prod_{n<m} s_n$. We let $\widehat{R}$ be the $\mathbb{S}$-adic completion of R. We will also assume that R is cotorsion-free (with respect to $\mathbb{S}$), this is to say that $\mathrm{Hom}(\widehat{R}, R) = 0$. More generally, an R-module M is $\mathbb{S}$-*cotorsion-free* if $\mathrm{Hom}_R(\widehat{R}, M) = 0$. We must say what it means if M has rank $\kappa \leq |R|$. (Note that R may not be a domain.) If $|M| > |R|$ it suffices to let $\mathrm{rk}(M) = |M|$. If $|M| \leq |R|$, then $\mathrm{rk}(M) = \kappa$ means that there is a free submodule $E = \bigoplus_{i<\kappa} Re_i$ of M such that M/E is $\mathbb{S}$-torsion. (Note that E also exists if $|M| > |R|$.) Recall that M is $\mathbb{S}$-torsion if for all $m \in M$ there is $s \in \mathbb{S}$ such that $sm = 0$. Similarly, a submodule N of M is $\mathbb{S}$-pure if $sM \cap N = sN$ for all $s \in \mathbb{S}$. If M is $\mathbb{S}$-torsion-free and $N \subseteq M$, then we denote by N_* the smallest pure submodule of M containing N, i.e. $N_* = \{m \in M | \exists s \in \mathbb{S} \text{ and } sm \in N\}$.

We will write $\mathrm{Hom}(M, N)$ for $\mathrm{Hom}_R(M, N)$ and in what follows all appearances of torsion, pure, etc. refer to $\mathbb{S}$ and we will therefore not mention the underlying set $\mathbb{S}$.

4.2 The construction

We recently proved in [14] the following result:

Theorem 4.1. *Let K be any torsion-free and reduced R-module of rank $\kappa < 2^{\aleph_0}$. Then there is a cotorsion-free R-module G of rank 3 if $\kappa = 1$, and of rank $3\kappa + 1$ if $2 \leq \kappa < 2^{\aleph_0}$, with submodule K such that $\mathrm{Hom}(G, K) = 0$ and $\mathrm{Hom}(G, G/K) = R\pi$ where $\pi : G \longrightarrow G/K$ ($g \mapsto g + K$) is the canonical epimorphism. In particular,*

$$0 \to K \to G \to G/K \to 0$$

is a cellular cover.

If K is a single copy of R, then the above result yields a particular cellular cover

$$0 \to R \to G \to H \to 0$$

where G and H are of rank 3 and 2, respectively. Thus R is the kernel of a cellular cover of rank two. However, if we choose $K = R \oplus R$, then Theorem 4.1 only proves that K is the kernel of a cellular cover of rank 7.
We want to push this bound down to 2 using a constructon that goes back to A.L.S. Corner (see also [14, Theorem 3.2]). As explained before, the case of free K is included and hence we have to assume that the rank of K is countable.

Theorem 4.2. *Let K be any cotorsion-free and reduced R-module of finite rank κ. Then there is a cellular exact sequence:*

$$0 \to K \to G \to H$$

where G is of rank $\kappa + 2$, and therefore H is of rank 2.

Proof. Let $E = \langle e_i : i \leq \kappa \rangle \subseteq K$ be a free-R-module, such that K/E is torsion. Choose $F = \langle f \rangle$ a free R-module on 1 generator. Let $C = K \oplus F$ and G be the following pure submodule of the completion $\widehat{C}$ of C:

$$G = \langle K, F, \sum_{i=1}^{\kappa} w_i e_i + wf \rangle_* \subseteq \widehat{C}$$

where w and w_i ($i < \kappa$) are elements in $\widehat{R}$ which are algebraically independent over $K \oplus F$. For their existence see [13, Theorem 1.1.20]. We claim that

$$0 \to K \to G \xrightarrow{\pi} H \to 0$$

is a cellular cover, where $\pi : \widehat{K} \oplus \widehat{F} \to \widehat{K}$ is the canonical projection. Clearly, the rank of G is then $\kappa + 2$, and the cokernel $H = \pi(G) = \langle f, wf \rangle_*$ has rank 2.

We first prove that $\widehat{K} \cap G = K$ and hence $ker(\pi \restriction_G) = K$. Let $x \in \widehat{K} \cap G$. Then there is $s \in \mathbb{S}$ such that

$$sx = k + f' + r(\sum_{i=1}^{\kappa} w_i e_i + wf) \in \widehat{K}$$

for some $r \in R$ and $k \in K, f' \in F$. It follows that

$$sx - k - r \sum_{i=1}^{\kappa} w_i e_i = f' + rwf$$

inside $\widehat{C}$. Since $\widehat{K} \cap \widehat{F} = 0$ we conclude that $f' + rwf = 0$ and by the algebraic independence of w also $r = 0$ and $f' = 0$. Thus

$$sx - k = 0$$

and so $sx = k \in K$ which implies $x \in K$ by the purity of K in $\widehat{K}$.

As in the proof of Theorem 3.2 in [14] we need to show that the group G satisfies the desired properties, namely $\mathrm{Hom}(G, K) = 0$, $\mathrm{Hom}(G, H) = \pi R$. Therefore let $\varphi \in \mathrm{Hom}(G, K)$. Thus

$$\varphi(\sum_{i=1}^{\kappa} w_i e_i + wf) = \sum_{i=1}^{\kappa} w_i \varphi(e_i) + w\varphi(f) \in K$$

by the unique lifting of φ to some map from $\widehat{C}$ to $\widehat{K}$ (again denoted by φ). Since the w_i and w were chosen algebraically independent over K we conclude that $\varphi(e_i) = \varphi(f) = 0$ for every $i \leq \kappa$. Note that $\varphi(e_i)$ and $\varphi(f)$ are also in K. Thus $\varphi \restriction_{E \oplus F} \equiv 0$. Since the quotient K/E is torsion this implies that also $\varphi(K) = 0$ by the torsion-freeness of K. Now, let $x \in G$, then there is $s \in S$ such that $sx \in \langle K, F, \sum_{i=1}^{\kappa} w_i e_i + wf \rangle$. It follows that $\varphi(sx) = 0$ and hence also $\varphi(x) = 0$ by torsion-freeness of K once more. Thus $\varphi \equiv 0$.
Now, let $\varphi \in \mathrm{Hom}(G, H)$. As above

$$s\varphi(\sum_{i=1}^{\kappa} w_i e_i + wf) = \sum_{i=1}^{\kappa} sw_i \varphi(e_i) + sw\varphi(f) = r_f f + r_w wf \in H$$

for some $s \in S$ and coefficients $r_f, r_w \in R$. By the algebraic independence of w_i and w we conclude that $\varphi(e_i) = 0$ for al $i \leq \kappa$ and hence K is in the kernel of φ. Moreover, letting $\varphi(f) = t_f f + t_w wf$ we conclude that

$$st_f wf + st_w w^2 f = r_f f + r_w wf$$

and thus $t_w = 0$ and $r_f = 0$. Hence $\varphi(f) = rf$ for some $r \in R$ and therefore φ factors through π and induces multiplication by r on H. We conclude that $\varphi = \pi r \in \pi R$ and this finishes the proof. $\square$

We conclude this paper with an open question.

Problem 4.3. *Can we also get a countable free group as the kernel of a rank two cellular cover?*

References

[1] A. K. Bousfield, *Homotopical localization of spaces*, Amer. J. Math. **119** (1997), 1321–1354.

[2] J. Buckner and M. Dugas, *Co-local subgroups of abelian groups*, pp. 29–37 in Abelian groups, rings, modules and homological algebra; Proceedings in honor of Enochs, Lect. Notes Pure Appl. Math. **249**, Chapman & Hill, Boca Raton, FL 2006.

[3] C. Casacuberta, *On structures preserved by idempotent transformations of groups and homotopy types. In: Crystallographic groups and their generalizations II* (Kortrijk, 1999), Contemp. Math. **262**, Amer. Math. Soc., Providence (2000), 39–69.

[4] C. Casacuberta, J. L. Rodríguez, and J.-Y. Tai, *Localizations of abelian Eilenberg–Mac Lane spaces of finite type*, preprint 1998.

[5] W. Chachólski, *On the functors CW_A and P_A,* Duke J. Math. **84** (1996), no. 3, 599–631.

[6] W. Chachólski, E. Dror Farjoun, R. Göbel, and Y. Segev, *Cellular covers of divisible abelian groups*, to appear in Proceedings of the third Arolla conference, Contemporary Math. AMS (2009).

[7] E. Dror Farjoun, *Cellular spaces, null spaces and homotopy localization,* Lecture Notes in Math. **1622** Springer-Verlag, Berlin–Heidelberg–New York 1996.

[8] E. Dror Farjoun, R. Göbel, Y. Segev, and S. Shelah, *On kernels of cellular covers*, Groups Geom. Dyn. **1** (2007), no. 4, 409–419.

[9] R. Flores, *Nullification and cellularization of classifying spaces of finite groups*, Trans. Amer. Math. Soc. **359** (2007), no. 4, 1791–1816.

[10] L. Fuchs, *Infinite Abelian Groups* – Vol. 1&2, Academic Press, New York (1970, 1973).

[11] L. Fuchs and R. Göbel, *Cellular covers of abelian groups*, Results in Mathematics (2008).

[12] R. Göbel and J. Trlifaj, *Approximation Theory and Endomorphism Algebras*, Expositions in Mathematics **41** Walter de Gruyter, Berlin (2006).

[13] R. Göbel and W. May, *Four submodules suffice for realizing algebras over commutative rings*, J. Pure Appl. Algebra **65** (1990), 29–43.

[14] R. Göbel, J. L. Rodríguez, and L. Strüngmann , *Cellular covers of cotorsion-free modules*, http://arxiv.org/abs/0906.4183, submitted.

[15] P. S. Hirschhorn: Model Categories and Their Localizations, *AMS Math. Surveys and Mon.*, vol. **99**, American Maths. Society, 2002.

[16] A. Nofech: *A*-cellular homotopy theories, *J. Pure Appl. Algebra*, **141** (1999), 249–267.

[17] J. L. Rodríguez and J. Scherer, *Cellular approximations using Moore spaces.* Cohomological methods in homotopy theory (Bellaterra, 1998), 357–374, Progr. Math., **196**, Birkhäuser, Basel, (2001).

[18] J. L. Rodríguez and J. Scherer, *A connection between cellularization for groups and spaces via two-complexes*, J. Pure Applied Algebra, **212** (2008), 1664–1673.

Lutz Strüngmann
Department of Mathematics,
University of Duisburg-Essen,
Campus Essen, 45117 Essen, Germany
e-mail: lutz.struengmann@uni-due.de

José L. Rodríguez
Área de Geometría y Topología,
Facultad de Ciencias Experimentales,
University of Almería,
La cañada de San Urbano, 04120 Almería, Spain
e-mail: jlrodri@ual.es